MATH 1010A/K 2017-18

University Mathematics

Tutorial Notes VI

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Question

(Q1) Suppose $f : [0,2] \to \mathbb{R}$ is the continuous function defined by

$$f(x) = \begin{cases} \left(x + \dfrac{1}{x}\right)^{-1} & \text{, when } 0 < x \leq 2 \\ a & \text{, when } x = 0 \end{cases}$$

- **(a)** Find all the critical point(s) of f in $(0,2)$.
- **(b)** For each critical point, identify whether it is a local maximum of f, local minimum of f, or neither.
- **(c)** Find the value of the constant a.
- **(d)** Find the absolute maximum and absolute minimum of f on $[0,2]$.

(Q2) Find the absolute extrema (if they exist) of the function $f : [-2,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^5 - 5x^4 + 5x^3 + 1 & \text{, when } -2 \leq x \leq 1 \\ \dfrac{1}{x-1} & \text{, when } 1 < x \leq 2 \end{cases}.$$

(Q3) Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) = \ln\left(\frac{1+|x|}{1+x^2}\right)$$

for any $x \in \mathbb{R}$.

- **(a)** Find $g'(x)$ if $x \neq 0$.
- **(b)** Find all local maximum point(s) (if any) of g.

(Q4) Define $f : \mathbb{R} \setminus \{\pm 1\} \to \mathbb{R}$ by

$$f(x) = \frac{x(x-2)}{\big||x|-1\big|}$$

for any $x \neq \pm 1$. Find all local minimum point(s) (if any) of f.

Answer

(A1) Note $f(x) = \left(x + \frac{1}{x}\right)^{-1}$ when $x \in (0,2)$ and $f(0) = 0$.

(a) Then $f'(x) = -\frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2} = \frac{\left(1 - x^2\right)}{\left(x^2 + 1\right)}$ for any $x \in (0,2)$.

If $f'(x) = 0$ for some $x \in (0,2)$, we can solve that $x = 1$ or $x = -1$ (rejected).

Hence, the only critical point of f on $(0,2)$ is $\left(1, f(1)\right) = \left(1, \frac{1}{2}\right)$.

(b) There are two method:

(Method 1) Note that

x	$0 < x < 1$	$x = 1$	$1 < x < 2$
$f'(x)$	+ve	0	-ve

and hence, $\left(1, \frac{1}{2}\right)$ is a local maximum by first derivative test.

(Method 2) Note that $f''(x) = \frac{-2x\left(x^2+1\right) - 2x\left(1 - x^2\right)}{\left(x^2+1\right)^2} = \frac{-4}{\left(x^2+1\right)^2}$ for any $x \in (0,2)$.

Then, $f''(1) = -1 < 0$ and

hence $\left(1, \frac{1}{2}\right)$ is a local maximum by second derivative test.

(c) Since f is continuous at $x = 0$, we have $\lim_{x \to 0^+} f(x) = f(0) = a$. That is

$$a = \lim_{x \to 0^+} \left(x + \frac{1}{x}\right)^{-1} = \lim_{x \to 0^+} \frac{x}{x^2 + 1} = 0.$$

(d) Note that f is continuous on $[0,2]$ and differentiable on $(0,2)$ with a critical point $\left(1, \frac{1}{2}\right)$.

Also note that $f(0) = a = 0$ and $f(2) = \frac{2}{5}$. By comparing the value of f at $0, 1, 2$,

we know that absolute maximum of f is $\frac{1}{2}$ occur at $x = 1$

and absolute minimum of f is 0 occur at $x = 0$.

(A2) Note that $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x-1} = +\infty$, so f has NO absolute maximum.

Note that

$$f'(x) = \begin{cases} 5x^4 - 20x^3 + 15x^2 & \text{, when } -2 < x < 1 \\ -\frac{1}{(x-1)^2} & \text{, when } 1 < x < 2 \end{cases}.$$

Solving $5x^2(x-1)(x-3) = 5x^4 - 20x^3 + 15x^2 = 0$ on $(-2,1)$,

we have $x = 0$, $x = 1$ (rejected) or $x = 3$ (rejected).

Note $-\frac{1}{(x-1)^2} = 0$ has no solution on $(1,2)$,

hence the only critical point of f is $\left(0, f(0)\right) = (0,1)$.

Compare $f(-2) = -151$, $f(0) = 1$, $f(1) = 2$ and $f(2) = 1$,

we know absolute minimum is -151 occur at $x = -2$.

(A3) Note that

$$g(x) = \ln\left(\frac{1 + |x|}{1 + x^2}\right) = \begin{cases} \ln\left(\frac{1+x}{1+x^2}\right) & \text{, when } x > 0 \\ \ln\left(\frac{1-x}{1+x^2}\right) & \text{, when } x < 0 \end{cases}.$$

(a) Then we have

$$g'(x) = \begin{cases} \dfrac{1+x^2}{1+x}\dfrac{(1+x^2)-2x(1+x)}{\left(1+x^2\right)^2} = \dfrac{1-2x-x^2}{(1+x)\left(1+x^2\right)} & \text{, when } x > 0 \\ \dfrac{1+x^2}{1-x}\dfrac{-(1+x^2)-2x(1-x)}{\left(1+x^2\right)^2} = \dfrac{-1-2x+x^2}{(1-x)\left(1+x^2\right)} & \text{, when } x < 0 \end{cases}.$$

(b) Solving $\dfrac{1-2x-x^2}{(1+x)\left(1+x^2\right)} = 0$ on $(0, +\infty)$,

we have $x = -1+\sqrt{2}$ or $x = -1-\sqrt{2}$ (rejected).

Solving $\dfrac{-1-2x+x^2}{(1-x)\left(1+x^2\right)} = 0$ on $(-\infty, 0)$,

we have $x = 1-\sqrt{2}$ or $x = 1+\sqrt{2}$ (rejected).

Note that we can have

x	$x < 1-\sqrt{2}$	$x = 1-\sqrt{2}$	$1-\sqrt{2} < x < 0$	$x = 0$
$g'(x)$	+ve	0	-ve	we do not know (even the existence)
x	$0 < x < -1+\sqrt{2}$	$x = -1+\sqrt{2}$	$x > -1+\sqrt{2}$	-
$g'(x)$	+ve	0	-ve	-

Note that g is continuous on $\mathbb{R}$ (why?) and differentiable on $\mathbb{R} \setminus \{0\}$.

By first derivative test, the only local minimum point of g is $\left(0, g(0)\right) = (0, 0)$.

(Q4) Note that

$$\begin{aligned} f(x) &= \frac{x(x-2)}{\left||x|-1\right|} \\ &= \begin{cases} \dfrac{x(x-2)}{|x|-1} & \text{, when } |x|-1 > 0 \\ \dfrac{x(x-2)}{1-|x|} & \text{, when } |x|-1 < 0 \end{cases} \\ &= \begin{cases} \dfrac{x(x-2)}{x-1} & \text{, when } x > 1 \\ -\dfrac{x(x-2)}{1+x} & \text{, when } 0 \le x < 1 \\ \dfrac{x(x-2)}{1-x} & \text{, when } -1 < x < 0 \\ \dfrac{x(x-2)}{1+x} & \text{, when } x < -1 \end{cases}. \end{aligned}$$

Then

$$f'(x) = \begin{cases} \dfrac{x^2+2x+2}{(x-1)^2} & \text{, when } x > 1 \\ \dfrac{3x^2-6x+2}{(x+1)^2} & \text{, when } 0 < x < 1 \\ -\dfrac{x^2+2x+2}{(x-1)^2} & \text{, when } -1 < x < 0 \\ -\dfrac{3x^2-6x+2}{(x+1)^2} & \text{, when } x < -1 \end{cases}$$

By solving $f'(x) = 0$ (please write the detail yourself), we can only get $x = 1 - \frac{1}{\sqrt{3}}$.

Then we draw the table

x	$x < -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1 - \frac{1}{\sqrt{3}}$	$x = \frac{1}{\sqrt{3}}$	$1 - \frac{1}{\sqrt{3}} < x < 1$	$x > 1$
$f'(x)$	-ve	-ve	don't know	+ve	0	-ve	+ve

Note f is continuous on $\mathbb{R} \setminus \{\pm 1\}$ and differentiable on $\mathbb{R} \setminus \{\pm 1, 0\}$.

By first derivative test, the only local minimum point of f is $\big(0, f(0)\big) = (0, 0)$.